Written Exam for the M.Sc. in Economics Autumn 2013 (Fall Term)

Financial Econometrics A: Volatility Modelling

Question A:

Solution A.1: In the sample $\tau+1, ..., T$, y_t is a Markov chain with a Gaussian transition density which satisfies the regularity conditions (to be verified) such that the drift criterion can be applied. To see this rewrite,

$$\sigma_t^2 = 1 - \alpha + \alpha \varepsilon_{t-1}^2 = 1 - \alpha + \alpha \left(y_{t-1}/v_{t-1}\right)^2.$$

Moreover, with $\delta(y) = 1 + y^2$, (note "$t+1$")

$$\begin{aligned} E\left(\delta\left(y_{t+1}\right)|y_t = y\right) &= 1 + (\omega+\gamma) E\left(\left(1 - \alpha + \alpha\left(y_t^2/(\omega+\gamma)\right)\right) z_{t+1}^2 | y_t = y\right) \\ &= 1 + (\omega+\gamma)(1-\alpha) + \alpha y^2, \end{aligned}$$

such that standard arguments give $0 \leq \alpha < 1$. Hence for $\omega + \gamma > 0$, and $\alpha < 1$ the process is geometrically ergodic with a stationary solution with $E\left(y_t^2\right) < \infty$. Finally, $E\left(y_t^2\right) = (\omega + \gamma)$.

Solution A.2: $L_T(\theta) = L_{T,1}(\theta) + L_{T,2}(\theta)$ with $\sigma_{1t}^2 = \omega(1-\alpha) + \alpha y_{t-1}^2$ (since $t \leq \tau$) and $\sigma_{2t}^2 = (\omega+\gamma)(1-\alpha) + \alpha y_{t-1}^2$ (since $t > \tau$)

$$L_{T,1} = \sum_{t=2}^{\tau}\left(\log\left(\sigma_{1t}^2\right) + y_t^2/\sigma_{1t}^2\right) \quad \text{and} \quad L_{T,2} = \sum_{t=\tau+2}^{T}\left(\log\left(\sigma_{2t}^2\right) + y_t^2/\sigma_{2t}^2\right).$$

The two-step: variance targeting. Hence, as $s_1^2 = \hat{\omega} + \hat{\gamma}$, and $s_2^2 = \hat{\gamma}$, we find $\hat{\omega} = s_1^2 - s_2^2$. Next $\hat{\alpha}$ is found by maximizing $L_T(\hat{\omega}, \hat{\gamma}, \alpha)$ over α (or simply an ARCH(1) model for the full sample). Less efficient, but clearly simpler.

Solution A.3: Simple differentiation and insertion gives the first part. Next, as $m_t := \left(1 - z_t^2\right)(1-\alpha_0) / \left((\omega_0+\gamma_0)(1-\alpha_0) + \alpha_0 y_{t-1}^2\right)$ is a martingale difference, $E(m_t) = 0$, and we use the CLT (applying the geometric ergodicity of y_t) to see the result (and $E\left(1 - z_t^2\right)^2 = 2$). That $\kappa < \infty$, follows by

$$(1-\alpha_0) / \left((\omega_0+\gamma_0)(1-\alpha_0) + \alpha_0 y_{t-1}^2\right) < 1/(\omega_0+\gamma_0) < \infty.$$

Solution Question A.3: The x_t series indeed looks as one would expect from the spARCH as around $\tau = 500$ the (conditional) volatility seems to change dramatically. This is confirmed in the graph where indeed the smooth curve indicates a one shift in volatility around $t = 500$. That $\hat{a} + \hat{b} = 1$ is the usual IGARCH issue (misspecified model potentially: room for some lines of explanation here). Normality is clearly rejected - hence misspecified model - fat tails etc. That there is no ARCH "left" from the test of ARCH effects is again stemming from the IGARCH filter discussion: a misspecifed IGARCH filters the (realized) volatility.

Question B:

Solution Question B.1: Simple application of drift criterion with $\delta(\sigma) = 1 + (\log \sigma^2)^2$ implies that $\sigma_t \in \mathbb{R}_+$ satisfies the assumptions of Theorem 1 in the SV lecture notes as $\rho^2 1 (|\log \sigma^2| \leq \gamma)$ tends to zero (is bounded by a constant) as $\log \sigma^2$ tends to infinity, whatever ρ^2 is. That $\rho = 1$ is allowed, may be explained by noting that as $\left|\log \sigma_{t-1}^2\right| \to \infty$, or 'drifts away', then $\delta_t \to 0$, and $\log \sigma_t^2$ reduces to Gaussian white noise $N\left(\mu, \sigma_\xi^2\right)$. In particular, $\rho = 10^9$ is also implying stationarity etc. Indeed the joint process is also weakly mixing, either by lecture notes theorem or quoting Meitz and Saikkonen (2008)

Solution Question B.2: $\varepsilon_t = \log z_t^2 - \mu$, with z_t Gaussian, and therefore ε_t is not. Know that $V(\log|z_t|) = \pi^2/8$ and hence $\sigma_\varepsilon^2 = V(2\log|z_t|) = \pi^2/2$.

The linear Kalman filter would not apply as even in the prediction step, we get

$$X_{t|t-1} = E\left(g\left(X_{t-1}\right) X_{t-1} | Y_{1:t-1}\right) \neq cE\left(X_{t-1} | Y_{1:t-1}\right) = cX_{t-1|t-1}.$$

In the lecture notes the notation, $X_{t|t-1} = \hat{x}_{t|t-1}$ is used.

Solution B.3: As noted the linear KF does not apply - neither does the extended KF seem promising as $\rho(x)$ is not differentiable. Would need some simulation based way, such as the particle filter to simulate the likelihood function. Discussion, and/or summary of particle filter estimation is needed here. Note that an alternative would be GMM but this is not covered in notes and letcures. The ox code piece is the Bootstrap proposal draws (details to be included in answer).

Solution Question B.4: TSV: clearly see the threshold. GARCH: higher spikes - explain. Explanations expected. An estimator of the variance given the past can be obtained from the predicted latent proces for log-volatility, $h_{t|t-1} = E(h_t | x_{t-1}, ..., x_1)$, which could be estimated as $\hat{h}_{t|t-1} = n^{-1} \sum_{i=1}^n h_t^{(i)}$, where $\{h_t^{(i)}\}_{i=1}^n$ denotes the particles from the prediction step, i.e. before resampling. The conditional variance is obtained as $\hat{\sigma}_{t,MLE}^2 = \exp(\hat{h}_{t|t-1})$. This is the predicted volatility. The corresponding filtered volatility is based on the filtered latent process, $h_{t|t-1} = E(h_t | x_t, ..., x_1)$, also conditioning on the current observed observation x_t. This is estimated using the weighted particles, $\hat{h}_{t|t} = \sum_{i=1}^{n} \hat{w}_t^{(i)} h_t^{(i)}$, or after resampling, $\hat{h}_{t|t} = n^{-1} \sum_{i=1}^{n} \tilde{h}_t^{(i)}$, where $\{\tilde{h}_t^{(i)}\}_{i=1}^{n}$ is obtained by resampling with replacement from the catagorical distribution given by the possible outcomes and corresponding probabilities, $(x_t^{(1)}, ..., x_t^{(n)}; w_t^{(1)}, ..., w_t^{(n)})$.